Problem orientable evaluations as L-subsets

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Neuchatel 26.6.2018

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Introduction

Suppose

- We consider evaluations as sets over a *lattice*.
- E.g.: G. Restrepo's evaluation of 40 refrigerants using *parameters ODP*, *GWP* and *ALT*, normalized so that the values are in $[0, 1]$.
- The lattice is $L = [0, 1]^3$, since with a refrigerant *ref* we associate the following triple of real numbers between 0 and 1:

$$(ODP(ref); GWP(ref); ALT(ref)) \in [0, 1]^3.$$

He obtained, for example,

refrigerants	*values of* $(ODP; GWP; ALT)$
ref_1	$(0,19607843; 0,31621622; 0,01406219)$
ref_2	$(0,16078431; 0,72432432; 0,0312497)$
ref_3	$(0,00980392; 0,12027027; 0,00374969)$
ref_4	$(0,00431373; 0,00513514; 0,00040594)$
$\vdots$	

The evaluation associates with ref_4 the *truth value*

$$tv(ref_4\ has\ (ODP; GWP; ALT))$$

$$= (0,00431373; 0,00513514; 0,00040594).$$

The entries of this table are in $L = [0,1]^3$.

In formal terms: this evaluation is a mapping

$$\mathcal{E}\colon \{ref_1, ref_2, \ldots\} \times \{(ODP; GWP; ALT)\} \to [0,1]^3,$$

with, e.g., the value

$$\mathcal{E}(ref_4, (ODP; GWP; ALT))$$

$$= (0,00431373; 0,00513514; 0,00040594).$$

The values are elements of the lattice $L = [0,1]^3$.

Consider such evaluations as L-subsets

The general case:

— An *evaluation* $\mathcal{E}$ of objects $o_i \in O$ w.r.t. attributes $a_k \in A$ and over L is a mapping

$$\mathcal{E}\colon O \times A \to L\colon (o_i, a_k) \mapsto \mathcal{E}((o_i, a_k)) = tv(o_i \text{ has } a_k),$$

— i.e. we consider an *L-subset* $\mathcal{E}$ of $O \times A$, containing (o_i, a_k) with the *truth value* $tv(o_i \text{ has } a_k) \in L$.

Example: Evaluations of objects o_i w.r.t. attributes a_k,

— Consider

$$L^{O \times A} := \{\mathcal{E} \mid \mathcal{E} \colon O \times A \to L\},$$

the set of all L–*subsets* of $O \times A$, for a given lattice L.

— In case $L = [0, 1]^3$, an L-subset of $O \times A$ is an association of a triple of parameter values to every element $(o, a) \in O \times A$.

We can *choose a set theory and its logic* over L on $L^{O \times A}$, allows problem–orientation.

— On *L–subsets* $\mathcal{S}, \mathcal{S}'$ of a set X we introduce *L–inclusion* as follows:

$$\mathcal{S} \subseteq_L \mathcal{S}' \iff \forall\, x \in X: \; \mathcal{S}(x) \leq \mathcal{S}'(x).$$

— Intersections of two such L-subsets can be defined, using *t-norms* $\tau : L \times L \to L$, mappings with *symmetry*, *monotony*, *associativity* and *side condition* $\tau(x, 1_L) = x$. They yield *τ–intersections* $\mathcal{I}$ on L^X:

$$\mathcal{I}(x) = (\mathcal{M} \cap_\tau \mathcal{N})(x) = \tau(\mathcal{M}(x), \mathcal{N}(x)).$$

The most important t–norms:

— The *standard* norm s is defined as

$$s(x, y) = x \wedge y.$$

— The *drastic* norm is

$$d(x, y) = \begin{cases} x & y = 1_L, \\ y & x = 1_L, \\ 0_L & \text{otherwise.} \end{cases}$$

— And if $L = [0, 1]$ there is the *algebraic product* a and the *bounded difference* b:

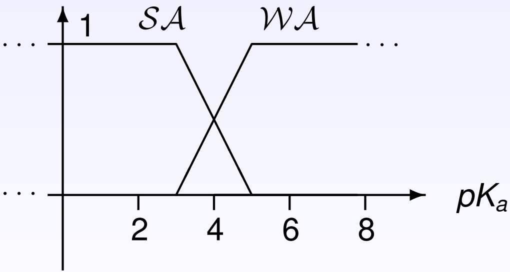
$$a(x, y) = x \cdot y, \; b(x, y) = \mathrm{Max}\{0, x + y - 1\}.$$

In particular the following is true:

$$d(x, y) \leq \tau(x, y) \leq s(x, y).$$

Example: Models $\mathcal{SA}$, $\mathcal{WA}$ of *strong* and *weak* acid

may look like that:



If we chose (e.g., in a possibly problem-oriented way) the t-norm $\tau = s$,

an acid with pK_a-value 4 is both strong and weak, while, if $\tau = d$, $(\mathcal{SA} \cap_d \mathcal{WA})(r) = 0$ (but $\mathcal{SA}(4) = \mathcal{WA}(4) = 0.5$).
We use a notion of truth, based on τ and its residuum:

— $\tau^\star : L \times L \to L$ is a *residuum* of τ, iff

$$\tau(x, y) \leq \nu \iff x \leq \tau^\star(y, \nu).$$

If $\tau(\alpha, \bigvee M) = \bigvee_{\beta \in M} \tau(\alpha, \beta)$ holds, then

$$\tau^\star(\alpha, \beta) = \bigvee \{\gamma \mid \tau(\alpha, \gamma) \leq \beta\}.$$

In this case τ is called a *residual t*–norm. *This yields a logic* corresponding to L and τ, namely $\tau^\star$.

Examples of residua for $L = [0, 1]$:

$$
\begin{aligned}
s^\star(\alpha, \beta) &= \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ \beta & \text{otherwise}, \end{cases} \\
d^\star(\alpha, \beta) &= \begin{cases} \beta & \text{if } \alpha = 1, \\ 1 & \text{otherwise}, \end{cases} \\
a^\star(\alpha, \beta) &= \begin{cases} \beta/\alpha & \text{if } \alpha \neq 0, \\ 1 & \text{otherwise}, \end{cases} \\
b^\star(\alpha, \beta) &= \mathrm{Min}\{1, 1 - \alpha + \beta\}.
\end{aligned}
$$

We have choices, can use a *problem orientation*

— Choose a suitable lattice L as set of values, pick a suitable residual t-norm τ obtaining a set theory. Its residuum $\tau^\star$ gives the corresponding logic. Apply that to $\mathcal{E} \in L^{O \times A}$, the evaluation considered, and get a basis of the implications (see below)!

Exploration

For the exploration of the evaluation $\mathcal{E}$ we can use that object o *has* attribute a if and only if $\mathcal{E}(o, a) > 0$. We put

$$\mathcal{A}'(o) = \tau^\star(\mathcal{A} \Rightarrow \mathcal{E}) = \bigwedge_{a \in A} \tau^\star(\mathcal{A}(a), \mathcal{E}(o, a)),$$

and we evaluate $\mathcal{A} \in L^A$ *implies* $\mathcal{B} \in L^A$ in $\mathcal{E}$ by:

$$\tau^\star(\mathcal{A} \Rightarrow \mathcal{B}) = \bigwedge_{o \in O} \tau^\star(\mathcal{A}'(o), \mathcal{B}'(o)).$$

$\mathcal{A} \Rightarrow \mathcal{B}$ *holds* in $\mathcal{E}$ if and only if $\tau^\star(\mathcal{A} \Rightarrow \mathcal{B}) = 1$, i.e., iff $\mathcal{A}' \subseteq_L \mathcal{B}'$. Defining *pseudo-contents*, by

$\mathcal{P} \neq \mathcal{P}''$ and for each pseudo-content $\mathcal{Q} \subset_L \mathcal{P}$: $\mathcal{Q}'' \subseteq_L \mathcal{P}$,

we get the *Duquenne/Guigues-basis which implies every attribute implication following from* $\mathcal{E}$,

$$\mathbb{P} = \{\mathcal{P} \Rightarrow (\mathcal{P}'' \setminus \mathcal{P}) \mid \mathcal{P} \text{ pseudo-content}\}.$$

Adding substructures and using simplified binary parameters $nODP^*$, $nGWP^*$, $nALT^*$, . . ., obtain:

$\mathcal{E}$	$nODP^*$	$nGWP^*$	$nALT^*$	nC	Cl	F	Br	I	$ether$	CO_2	NH_3
1	1	0	0	0	1	1	0	0	0	0	0
2	0	1	0	0	1	1	0	0	0	0	0
6	0	0	0	1	1	1	0	0	0	0	0
7	0	0	0	1	1	1	0	0	0	0	0
8	0	1	1	0	0	1	0	0	0	0	0
16	0	0	0	1	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	1	0
22	1	0	0	0	1	1	1	0	0	0	0
23	0	1	1	1	0	1	0	0	0	0	0
29	0	1	1	1	0	1	0	0	1	0	0
32	0	0	0	0	1	0	0	0	0	0	0
33	1	0	0	1	1	1	0	0	0	0	0
35	1	0	1	1	1	1	0	0	0	0	0
36	0	0	0	0	0	1	0	1	0	0	0
37	0	0	0	1	0	0	0	0	1	0	0
38	0	0	0	0	0	0	0	0	0	0	1
39	0	0	0	1	0	1	0	0	1	0	0
40	0	0	0	1	0	1	0	0	1	0	0

The Duquenne/Guigues basis of it yields all what follows,

it can be obtained online, using CONEXP–1.3.

$$\begin{aligned}
\{nODP^*\} &\implies \{Cl, F\} \\
\{nGWP^*\} &\implies \{F\} \\
\{nALT^*\} &\implies \{F\} \\
\{nC, Cl\} &\implies \{F\} \\
\{nALT^*, Cl, F\} &\implies \{nODP^*, nC\} \\
\{nGWP^*, nC, F\} &\implies \{nALT^*\} \\
\{Br\} &\implies \{nODP^*, Cl, F\} \\
\{I\} &\implies \{F\} \\
\{ether\} &\implies \{nC\} \\
\{nALT^*, nC, F, ether\} &\implies \{nGWP^*\}
\end{aligned}$$

Summarizing we obtain:

In order to explore your evaluation of objects $o \in O$ according to given attributes $a \in A$ do the following:

- Choose a suitable set theory, i.e. a residual τ and its $\tau^\star$,
- use Brüggemann's PyHasse in order to obtain partial orders and their visualizations by Hasse diagrams,
- evaluate, using CONEXP if it is binary, the Duquenne/Guigues basis is a set of *hypotheses* on possibly interesting bigger sets $\Omega \supset O$ of objects. Try to prove (or at least to check) these!

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